Fourier Transform

If f(t) be the function of 't' then fourier transform of f(t) is defined as

$$F[f(t)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt = F(s) \quad \text{—— (1)}$$

Fourier sine transform

$$F_s\, f(t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t) \sin st\, dt \quad \text{—— (2)}$$

Fourier cosine transform

$$F_c\, f(t) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t) \cos st\, dt \quad \text{—— (3)}$$

Fourier Transform

Fourier inverse transform / Inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds. \quad \text{—— (4)}$$

Inv fourier sine transform

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F(s) \sin sx\, ds \quad \text{—— (5)}$$

Inverse Fourier cosine transform :

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F(s) \cos sx\, dx \quad \text{—— (6)}$$

Inverse Fourier Transform

Q.1. Find the fourier transform of

$$f(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a. \end{cases}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{0} f(x)\, e^{isx}\, dx + \int_{0}^{a} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{0} 1 \cdot e^{isx}\, dx + 0$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \frac{e^{isx}}{is} \Big|_{-a}^{0} = \frac{1}{\sqrt{2\pi}} \frac{1}{(is)} \left[e^{ias} - e^{-ias} \right]$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \frac{2}{s} \left[\frac{e^{ias} - e^{-ias}}{2i} \right]$$

$$\left[\because e^{i\theta} = \cos\theta + i\sin\theta \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2}{s} \sin sa$$

$$F[f(x)] = \underbrace{\sqrt{\frac{2}{\pi}}\, \frac{\sin sa}{s} = F(s)}_{\text{Ans}}$$

Find the fourier Transform of $\bar{e}^{ax^2}$, $a>0$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{e}^{ax^2} \cdot e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{e}^{ax^2 - isx} dx \qquad 2 \times \sqrt{a} x \times \frac{is}{2\sqrt{a}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{e}^{\left(ax^2 - 2 \times \sqrt{a} x \times \frac{is}{2\sqrt{a}} + \left(\frac{is}{2\sqrt{a}}\right)^2\right) + \left(\frac{is}{2\sqrt{a}}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{e}^{\left(x\sqrt{a} - \frac{is}{2\sqrt{a}}\right)^2} \cdot e^{-\frac{s^2}{4a^2}} dx.$$

putting $x\sqrt{a} - \frac{is}{2\sqrt{a}} = y$

$\sqrt{a}\, dx - 0 = dy$

$dx = \frac{1}{\sqrt{a}} dy$ $\qquad \left[\because \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}\right]$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{e}^{y^2} \cdot e^{-s^2/4a} \frac{dy}{\sqrt{a}}$$

$$\underbrace{\bar{e}^{s^2/4a} \cdot \frac{1}{\sqrt{a}\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{e}^{y^2} dy} = \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{a}} \cdot \bar{e}^{\frac{s^2}{4a}}$$

$$F[f(x)] = \frac{e^{-s^2/4a}}{\sqrt{2a}} \quad \text{Ans}$$

Properties of Fourier Transforms

① Linear property

If $F_1(s)$ & $F_2(s)$ are fourier transforms of $f_1(x)$ & $f_2(x)$ respectively then

$$F[a f_1(x) + b f_2(x)] = a F_1(s) + b F_2(s)$$

② Change of scale property

If $F(s)$ be the fourier transform of $f(x)$ then

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

③ Shifting property

If $F(s)$ is fourier transform of $f(x)$ then

$$F[f(x-a)] = e^{isa} F(s)$$

④ Modulation theorem

If $F(s)$ is the fourier transform of $f(x)$, then

$$F[f(x) \cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$$

⑤ If $F[f(x)] = F(s)$ then

$$F\{x^n f(x)] = (-i)^n \frac{d^n}{ds^n} F(s)$$

⑥ $F\{f'(x)\} = isF(s)$ if $f(x) \to 0$ as $x \to \pm\infty$

⑦

$$F\left\{\int_a^x f(x)\,dx\right\} = \frac{F(s)}{(-is)}$$

<u>Note</u>: $F_s(s)$ & $F_c(s)$ be the Fourier Sine & Cosine transform of $f(x)$ respectively then

① $F_s\{af(x)+b\,g(x)\} = a\,F_s[f(x)\} + b\,F_s\{g(x)\}$

② $F_c\{af(x)+b\,g(x)\} = a\,F_c\{f(x)\} + b\,F_c\{g(x)\}$

③ $F_s\{f(ax)\} = \frac{1}{a} F_s\left(\frac{s}{a}\right)$

④ $F_c\{f(ax)\} = \frac{1}{a} F_c\left(\frac{s}{a}\right)$

⑤ $F_s\{f(x)\sin ax\} = \frac{1}{2}[F_c(s+a) - F_c(s+a)]$

⑥ $F_c\{f(x)\sin ax\} = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

⑦ $F_s\{f(x)\cos ax\} = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

<u>Convolution</u>

The convolution of two functions $f(x)$ & $g(x)$ is defined as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\,g(x-u)\,du$$

mostly even odd functions integrals

$\int_{-a}^{a} f(x) = \begin{cases} 2f(x), & \text{even} \\ 0 & \text{odd} \end{cases}$

EXERCISE 45.2

1. Find the Fourier Transform of $f(x)$ if

$$f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$$

Ans. $\frac{1}{\sqrt{2\pi}} \frac{2i}{s^2}(as\cos as - \sin as)$

2. Show that the Fourier Transform of

$$f(x) = \begin{cases} a - |x| & \text{for } |x| < a \\ 0 & \text{for } |x| > a > 0 \end{cases}$$

is $\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos as}{s^2}\right)$.

Hence show that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$

3. Show that the Fourier Transform of

$$f(x) = \begin{cases} \frac{\sqrt{2\pi}}{2a} & \text{for } |x| \le a \\ 0 & \text{for } |x| > a \end{cases}$$

is $\frac{\sin sa}{sa}$

4. Find Fourier Transform of $e^{-a|x|}$ if $a > 0$ and $x > 0$. **Ans.** $\frac{2a}{a^2+s^2}$

5. Find Fourier Transform of $\frac{1}{\sqrt{|x|}}$.

6. Find the Fourier Transform of $f(x) = \begin{cases} e^{ikx}, & a < x < b \\ 0, & x < a \text{ and } x > b \end{cases}$ **Ans.** $\frac{i}{\sqrt{2\pi}(k+s)}\left[e^{i(k+s)a} - e^{i(k+s)b}\right]$

7. Show that the Fourier Transform of

$$f(x) = \begin{cases} 0 & \text{for } x < \alpha \\ 1 & \text{for } \alpha < x < \beta \\ 0 & \text{for } x > \beta \end{cases}$$

is $\frac{1}{\sqrt{2\pi}} \frac{2i}{s^2}(as \cos as - \sin as)$

8. If $F(s)$ is the Fourier Transform of $f(x)$, prove that

$$f[e^{iax} f(x)] = F(s+a)$$

9. Find Fourier transform of

$$F(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a \end{cases}.$$

Ans. $\left(\frac{2a^2}{s} - \frac{4}{s^3}\right)\sin as + \frac{4a}{s^2}\cos as$

10. Show that the Fourier Sine Transform of $f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 < x < 2 \\ 0 & \text{for } x > 2 \end{cases}$ is $\frac{2\sin s(1-\cos s)}{s^2}$.

11. Show that the Fourier Sine Transform of $\frac{x}{1+x^2}$ is $\sqrt{\frac{\pi}{2}}\, a s\, e^{-as}$.

12. Find Fourier Sine Transform of

$$f(x) = \frac{1}{x(x^2+a^2)}$$

(U.P.T.U. 2001) **Ans.** $\frac{\pi}{2a^2}(1-e^{-as})$

Fourier Series:

A periodic func^n $f(x)$ can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$\downarrow$

is known as fourier series expasion of $f(x)$.

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

a_0, a_n, b_n are called as fourier co-efficient

Note : To get similar formula of a_0, $\frac{1}{2}$ has been written with a_0 in Fourier series.

Example 1. *Find the Fourier series representing*

$$f(x) = x, \quad 0 < x < 2\pi$$

and sketch its graph from $x = -4\pi$ *to* $x = 4\pi$.

Solution. Let $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + ... + b_1 \sin x + b_2 \sin 2x + ...$ ...(1)

Hence $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

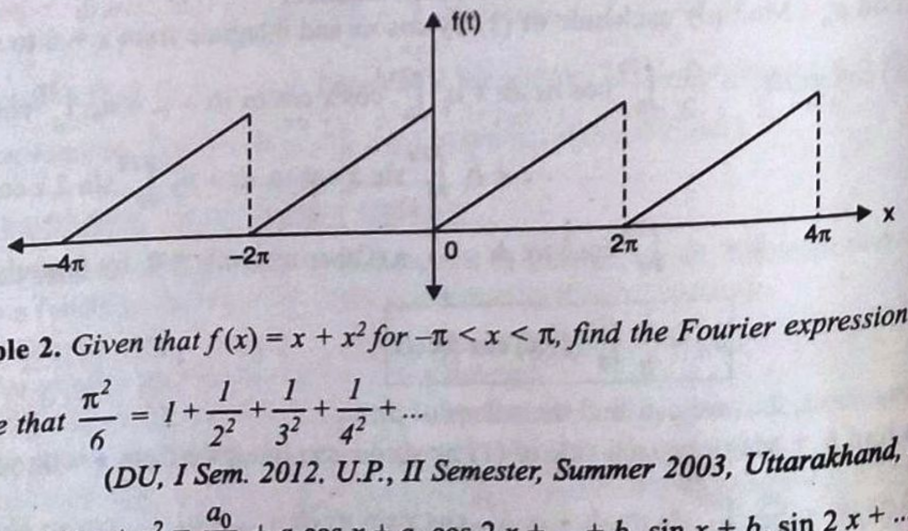
$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - 1.\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1.\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{-2\pi\cos 2n\pi}{n}\right] = -\frac{2}{n}$$

Substituting the values of $a_0, a_1, a_2 ..., b_1, b_2 ...$ in (1), we get

$$x = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + ...\right]$$ **Ans.**



Example 2. *Given that* $f(x) = x + x^2$ *for* $-\pi < x < \pi$, *find the Fourier expression of* $f(x)$.

Deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$

(DU, I Sem. 2012. U.P., II Semester, Summer 2003, Uttarakhand, June 2009)

Solution. Let $x + x^2 = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + ... + b_1\sin x + b_2\sin 2x + ...$ ...(1)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \int_{-\pi}^{\pi}(x + x^2)\cos nx\,dx$$

CHAPTER

19

MAXIMA AND MINIMA OF FUNCTIONS (TWO VARIABLES)

19.1 MAXIMUM VALUE

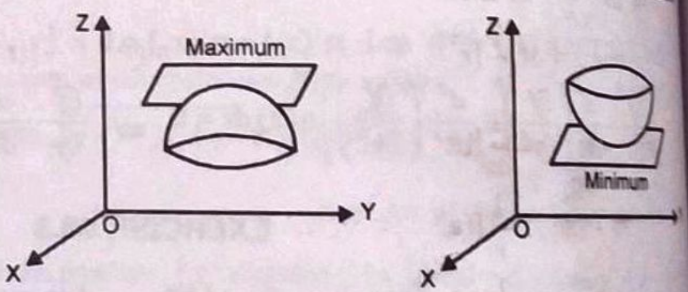
A function $f(x, y)$ is said to have a maximum value at $x = a$, $y = b$, if there exists a small neighbourhood of (a, b) such that,

$$f(a, b) > f(a + h, b + k)$$

Minimum Value. A function $f(x, y)$ is said to have a minimum value for $x = a, y = b$, if there exists a small neighbourhood of (a, b) such that.

$$f(a, b) < f(a + h, b + k)$$

The maximum and minimum values of a function are also called extreme or extremum values of the function.



Saddle Point or Minimax

It is a point where function is neither maximum nor minimum.

Geometrically such a surface (looks like the leather seat on the back of a horse) forms a ridge rising in one direction and falling in another direction.

19.2 CONDITIONS FOR EXTREMUM VALUES

If $f(a + h, b + k) - f(a, b)$ remains of the same sign for all values (positive or negative) of h, k then $f(a, b)$ is said to be extremum value of $f(x, y)$ at (a, b)

(*i*) If $f(a + h, b + k) - f(a, b) < 0$, then $f(a, b)$ is maximum.

(*ii*) If $f(a + h, b + k) - f(a, b) > o$, then $f(a, b)$ is minimum.

By Taylor's Theorem

$$f(a+h, b+k) = f(a, b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{(a,b)} + \frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right] + \ldots$$

$$\Rightarrow \quad f(a + h, b + k) - f(a, b) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{(a,b)} + \frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right] + \ldots$$

$$\Rightarrow \quad f(a + h, b + k) - f(a, b) = \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{(a,b)}$$

For small values of h, k, the second and higher order terms are still smaller and hence may be neglected.

The sign of L.H.S. of (2) is governed by $h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}$ which may be positive or negative depending on h, k.

Hence, the necessary condition for (a, b) to be a maximum or minimum is that

$$\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right) = 0 \Rightarrow \frac{\partial f}{\partial x}=0, \quad \frac{\partial f}{\partial y}=0$$

By solving the equations, we get, point $x = a$, $y = b$ which may be maximum or minimum value.

Then from (1)

$$f(a + h, b + k) - f(a, b) = \frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x \partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right]$$

$$= \frac{1}{2!}[h^2 r+2hks+k^2 t] \qquad ...(3)$$

where $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$ at (a, b)

Now the sign of L.H.S. of (3) is sign of $[rh^2 + 2hks + k^2 t]$

$$= \text{sign of } \frac{1}{r}[r^2h^2+2hkrs+k^2rt] = \text{sign of } \frac{1}{r}[(r^2h^2+2hkrs+k^2s^2)+(-k^2s^2+k^2rt)]$$

$$= \text{sign of } \frac{1}{r}[(hr+ks)^2+k^2(rt-s^2)]$$

$$= \text{sign of } \frac{1}{r}[(\text{always } + ve) + k^2(rt-s^2)] \qquad [(hr+ks)^2 = +ve]$$

$$= \text{sign of } \frac{1}{r}[k^2(rt-s^2)] = \text{sign of } r \text{ if } rt - s^2 > 0$$

Hence, if $rt - s^2 > 0$, then $f(x, y)$ has a maximum or minimum at (a, b) according as $r < 0$ or $r > 0$.

Note: (*i*) If $rt - s^2 < 0$, then L.H.S. will change with h and k hence there is no maximum or minimum at (a, b), *i.e.*, it is a saddle point.

(*ii*) If $rt - s^2 = 0$, then $rh^2 + 2shk + tk^2 = \frac{1}{r}[(rh+sk)^2 + k^2(rt-s^2)]$

$$= \frac{1}{r}(rh+sk)^2 \text{ which is zero for values of } h, k, \text{ such that}$$

$$\Rightarrow \quad \frac{h}{k} = -\frac{s}{r}$$

This is, therefore, a doubtful case, further investigation is required.

9.3 WORKING RULE TO FIND EXTREMUM VALUES (D.U., April 2010)

(*i*) Differentiate $f(x, y)$ and find out

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$$

(*ii*) Put $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ and solve these equations for x and y. Let (a, b) be the values of (x, y).

(*iii*) Evaluate $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ for these values (a, b).

(iv) If $rt - s^2 > 0$ and

(a) $r < 0$, then $f(x, y)$ has a maximum value.

(b) $r > 0$, then $f(x, y)$ has a minimum value.

(v) If $rt - s^2 < 0$, then $f(x, y)$ has no extremum value at the point (a, b).

(iv) If $rt - s^2 = 0$, then the case is doubtful and needs further investigation.

Note: The point (a, b), which are the roots of $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ are called stationary points.

Example 1. *Find the absolute maximum and minimum values of*

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on triangular plane in the first quadrant, bounded by the lines $x = 0$, $y = 0$ and $y = 9 - x$. (Gujarat, I Semester, Jan. 20..)

Solution. We have, $f(x, y) = 2 + 2x + 2y - x^2 - y^2$

$$\frac{\partial f}{\partial x} = 2 - 2x \qquad \frac{\partial f}{\partial y} = 2 - 2y$$

$$\frac{\partial^2 f}{\partial x^2} = -2, \qquad \frac{\partial^2 f}{\partial x \partial y} = 0, \qquad \frac{\partial^2 f}{\partial x^2} = -2$$

For maxima and minima,

$$\frac{\partial f}{\partial x} = 0 \quad \Rightarrow \quad 2 - 2x = 0 \quad \Rightarrow \quad x = 1$$

and $$\frac{\partial f}{\partial y} = 0 \quad \Rightarrow \quad 2 - 2y = 0 \quad \Rightarrow \quad y = 1$$

At (1, 1) $$rt - s^2 = (-2)(-2) - 0 = +4.$$

Here $$r = \frac{\partial^2 f}{\partial x^2} = -2 = -\text{ve}$$

Hence $f(x, y)$ is maximum at (1, 1).

Maximum value of $f(x, y) = 2 + 2 + 2 - 1 - 1 = 4$.

Example 2. *Examine the function $f(x, y) = y^2 + 4xy + 3x^2 + x^3$ for extreme values.* (M.U. 20..)

Solution. We have, $f(x, y) = y^2 + 4xy + 3x^2 + x^3$

$$p = \frac{\partial f}{\partial x} = 4y + 6x + 3x^2 \qquad q = \frac{\partial f}{\partial y} = 2y + 4x$$

$$r = \frac{\partial^2 f}{\partial x^2} = 6 + 6x \qquad s = \frac{\partial^2 f}{\partial x \partial y} = 4$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2$$

For maxima or minima

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

$\Rightarrow \quad 4y + 6x + 3x^2 = 0 \quad ...(1)$

$\Rightarrow \quad 2y + 4x = 0$

$y = -2x$

Putting the value of y from (2) in (1), we get

$$4(-2x) + 6x + 3x^2 = 0$$

$$\Rightarrow \quad 3x^2 + 6x - 8x = 0$$

$$\Rightarrow \quad 3x^2 - 2x = 0 \quad \Rightarrow \quad x(3x - 2) = 0$$

$$x = 0 \text{ or } x = \frac{2}{3}$$

$\Rightarrow$

when $x = 0$ then $y = 0$

when $x = \frac{2}{3}$ then $y = -2\left(\frac{2}{3}\right) = -\frac{4}{3}$

Thus, the stationary points are (0, 0) and $\left(\frac{2}{3}, \frac{-4}{3}\right)$,

	(0, 0)	$\left(\frac{2}{3}, \frac{-4}{3}\right)$
$r = 6 + 6x$	6	10
$s = 4$	4	4
$t = 2$	2	2
$rt - s^2$	-4	$+4$

At (0, 0) there is no extremum value, since $rt - s^2 < 0$;

At $\left(\frac{2}{3}, \frac{-4}{3}\right)$, $rt - s^2 > 0$, $r > 0$.

Therefore $\left(\frac{2}{3}, \frac{-4}{3}\right)$ is a point of minimum value.

The minimum value of $f\left(\frac{2}{3}, -\frac{4}{3}\right) = \left(\frac{-4}{3}\right)^2 + 4\left(\frac{2}{3}\right)\left(-\frac{4}{3}\right) + 3\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3$

$$\Rightarrow \quad f\left(\frac{2}{3}, \frac{-4}{3}\right) = \frac{16}{9} - \frac{32}{9} + \frac{12}{9} + \frac{8}{27} = \frac{8}{27} - \frac{4}{9} = -\frac{4}{27}$$ **Ans.**

Example 3. *Show that the minimum value of* $u = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$ *is* $3a^2$.

(M. U. 2002)

Solution. We have,

$$f(x, y) = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$$

$$p = \frac{\partial f}{\partial x} = y - \frac{a^3}{x^2} \qquad q = \frac{\partial f}{\partial y} = x - \frac{a^3}{y^2}$$

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{2a^3}{x^3} \qquad s = \frac{\partial^2 f}{\partial x \partial y} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{2a^3}{y^3}$$

For maxima and minima

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow \quad y - \frac{a^3}{x^2} = 0 \qquad \Rightarrow \quad x - \frac{a^3}{y^2} = 0$$

$$\Rightarrow \quad x^2 y = a^3 \quad \ldots(1) \qquad \Rightarrow \quad x = \frac{a^3}{y^2} \quad \ldots(2)$$

Putting the value of x from (2) in (1), we get

$$\left(\frac{a^3}{y^2}\right)^2 y = a^3 \Rightarrow \frac{a^6}{y^3} = a^3 \Rightarrow y = a \quad \ldots(3)$$